Chapter Seven
The Quantum Mechanical Simple Harmonic Oscillator

Introduction
The potential energy function for a classical, simple harmonic oscillator is given by
\[ V(x) = \frac{1}{2}kx^2 \]
where \( k \) is the spring constant. Such a classical oscillator has an angular frequency \( \omega = \sqrt{k/m} \), where \( m \) is the mass of the oscillator. Writing the potential energy function in terms of the angular frequency, rather than the spring constant, gives
\[ V(x) = \frac{1}{2}m\omega^2x^2 \quad (7.1) \]
The time-independent Schrodinger equation can then be written in the form
\[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \frac{1}{2}m\omega^2x^2 \psi(x) = E \psi(x) \quad (7.2) \]
or, in operator notation,
\[ \frac{\hat{\mathbf{p}}^2}{2m} \psi(x) + \frac{1}{2}m\omega^2 \hat{x}^2 \psi(x) = E\psi(x) \quad (7.3) \]
The potential energy function for this problem is different from the ones we have considered thus far. This potential energy function essentially spans all space. In the diagram below, you can see that there are regions where the total energy is greater than the potential (regions where the wave function would be sinusoidal) and regions where the total energy is less than the potential (regions where proper wave functions will be exponentially decreasing). Thus, a proper solution to the simple harmonic oscillator problem will be sinusoidal in the central region, but must be exponentially decreasing as \( x \) tends toward both positive and negative infinity.

![The Simple Harmonic Oscillator Potential](image)

**Fig. 7.1** The Simple Harmonic Oscillator Potential
Analytical Method

We begin with the time-independent Schrodinger equation for the simple harmonic oscillator, expressed in terms of the angular frequency

\[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \frac{1}{2} m\omega^2 x^2 \psi(x) = E \psi(x) \]  

(7.4)

In order to simplify this equation somewhat, we will let

\[ x = \alpha \xi \]  

(7.5)

This introduces an arbitrary constant \( \alpha \) which can then be defined in such a way as to create a dimensionless equation. The Schrodinger equation becomes

\[ \frac{-\hbar^2}{2m} \alpha^2 \frac{\partial^2}{\partial \xi^2} \psi(\xi) + \frac{1}{2} m\omega^2 \alpha^2 \xi^2 \psi(\xi) = E \psi(\xi) \]  

(7.6)

Multiplying by \(-2m\alpha^2/\hbar^2\) gives

\[ \frac{\partial^2}{\partial \xi^2} \psi(\xi) - \frac{m^2 \omega^2 \alpha^4}{\hbar^2} \xi^2 \psi(\xi) = -\frac{2m\alpha^2}{\hbar^2} E \psi(\xi) \]  

(7.7)

Now using our freedom to define \( \alpha \), we choose \( \alpha \) to be

\[ \alpha = \frac{\sqrt{\hbar/m\omega}}{} \]  

(7.8)

so that our equation simplifies to the dimensionless form

\[ \frac{\partial^2}{\partial \xi^2} \psi(\xi) - \xi^2 \psi(\xi) = -\frac{2}{\hbar\omega} E \psi(\xi) \]  

(7.9)

You can tell this equation is indeed dimensionless, since the energy has units of \( \hbar\omega \).

Letting \( K = 2E/\hbar\omega \) our equation reduces to

\[ \frac{\partial^2}{\partial \xi^2} \psi(\xi) = (\xi^2 - K) \psi(\xi) \]  

(7.10)

Note that

For large \( \xi \) (i.e., large displacements from the equilibrium position where \( \xi \to \infty \)) we expect the wave function to have the approximate form

\[ \psi(\xi) \approx e^{-\beta \xi^2} \]  

(7.11)

since we know that the wave function must tend toward zero for both positive and negative values of \( \xi \). Let's take this approximate form of the wave function as a trial
solution to our differential equation and see how well it satisfies our differential equation.

\[
\frac{\partial^2}{\partial^2 \xi} e^{-\beta \xi^2} = \frac{\partial}{\partial \xi} \left[ e^{-\beta \xi^2} \left( -2\beta \xi \right) \right] = e^{-\beta \xi^2} \left( 4\beta^2 \xi^2 \right) - e^{-\beta \xi^2} (2\beta) = 4\beta^2 e^{-\beta \xi^2} \left( \xi^2 - \frac{1}{2\beta} \right) \tag{7.12}
\]

Notice that if we set \( \beta = \frac{1}{2} \), this last equation reduces to

\[
\frac{\partial^2}{\partial^2 \xi} e^{-\frac{1}{2} \xi^2} = (\xi^2 - 1) e^{-\frac{1}{2} \xi^2} \tag{7.13}
\]

which is almost exactly the form of our dimensionless oscillator equation except for the energy constant \( K \). We expect the exact solution of our differential equation, therefore, to be of the form

\[
\psi(\xi) = h(\xi) e^{-\frac{1}{2} \xi^2} \tag{7.14}
\]

and hope that pulling out the exponential term will help us find a simple expression for \( h(\xi) \). (One can always hope!)

Our dimensionless differential equation for the quantum oscillator is

\[
\frac{\partial^2}{\partial \xi^2} \psi(\xi) = (\xi^2 - K) \psi(\xi) \tag{7.15}
\]

If we assume that the correct solution for this equation has the form of equation 7.14, we need to evaluate the second partial of this function. The first partial is given by

\[
\frac{\partial}{\partial \xi} \left( h(\xi) e^{-\frac{1}{2} \xi^2} \right) = \left( \frac{\partial}{\partial \xi} h(\xi) \right) e^{-\frac{1}{2} \xi^2} + h(\xi) \left( -\xi e^{-\frac{1}{2} \xi^2} \right) = \left( \frac{\partial h(\xi)}{\partial \xi} - \xi h(\xi) \right) e^{-\frac{1}{2} \xi^2} \tag{7.16}
\]

Taking the partial again, we obtain

\[
\frac{\partial^2}{\partial \xi^2} \psi(\xi) = \frac{\partial^2 h(\xi)}{\partial \xi^2} e^{-\frac{1}{2} \xi^2} + \frac{\partial h(\xi)}{\partial \xi} \left( -\xi e^{-\frac{1}{2} \xi^2} \right) + \frac{\partial h(\xi)}{\partial \xi} \left( -\xi e^{-\frac{1}{2} \xi^2} \right) + h(\xi) \left( - e^{-\frac{1}{2} \xi^2} - \xi \left( -e^{-\frac{1}{2} \xi^2} \right) \right) \tag{7.17}
\]

\[
\frac{\partial^2}{\partial \xi^2} \psi(\xi) = e^{-\frac{1}{2} \xi^2} \left\{ \frac{\partial^2 h(\xi)}{\partial \xi^2} - 2\xi \frac{\partial h(\xi)}{\partial \xi} + h(\xi) \left( \xi^2 - 1 \right) \right\}
\]

Our dimensionless differential equation for the quantum oscillator becomes

\[
e^{-\frac{1}{2} \xi^2} \left\{ \frac{\partial^2 h(\xi)}{\partial \xi^2} - 2\xi \frac{\partial h(\xi)}{\partial \xi} + h(\xi) \left( \xi^2 - 1 \right) \right\} = (\xi^2 - K) h(\xi) e^{-\frac{1}{2} \xi^2} \tag{7.18}
\]
This leads to a differential equation for the function \( h(\xi) \), since the exponential drops out, given by

\[
\frac{\partial^2 h(\xi)}{\partial \xi^2} - 2\xi \frac{\partial h(\xi)}{\partial \xi} + (K - 1)h(\xi) = 0
\]  

(7.19)

This equation we now solve using the assumption that the solution can be expressed in terms of a power series in \( \xi \) of the form

\[
h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j\xi^j
\]  

(7.20)

Differentiating the series term by term, once, gives

\[
\frac{\partial h(\xi)}{\partial \xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + \cdots = \sum_{j=0}^{\infty} ja_j\xi^{j-1}
\]  

(7.21)

Differentiating again gives

\[
\frac{\partial^2 h(\xi)}{\partial \xi^2} = (1 \cdot 2)a_2 + (2 \cdot 3)a_3\xi + (3 \cdot 4)a_4\xi^2 + \cdots = \sum_{j=0}^{\infty} (j+1)(j+2)a_{j+2}\xi^j
\]  

(7.22)

Substituting this into the differential equation for \( h(\xi) \) gives

\[
\sum_{j=0}^{\infty} (j+1)(j+2)a_{j+2}\xi^j - 2\xi\sum_{j=0}^{\infty} ja_j\xi^{j-1} + (K - 1)\sum_{j=0}^{\infty} a_j\xi^j = 0
\]  

(7.23)

Collecting terms of the same power and rearranging finally gives

\[
\sum_{j=0}^{\infty} (j+1)(j+2)a_{j+2}\xi^j - 2ja_j\xi^j + (K - 1)a_j\xi^j = 0
\]  

(7.24)

or

\[
\sum_{j=0}^{\infty} [(j+1)(j+2)a_{j+2} - 2ja_j + (K - 1)a_j]\xi^j = 0
\]  

(7.25)

Now if the differential equation must be valid for all values of the variable \( \xi \), then the only way this last equation can be valid for any arbitrary choice of \( \xi \) is for the coefficients to be identically zero, or

\[
(j + 1)(j + 2)a_{j+2} - 2ja_j + (K - 1)a_j = 0
\]  

(7.26)
Now this equation relates the coefficient $a_{j+2}$ to the coefficient $a_j$ according to the equation

$$a_{j+2} = \frac{2j + 1 - K}{(j + 1)(j + 2)} a_j$$  \hspace{1cm} (7.27)

This recursion formula enables us to write any coefficient if we know just two, $a_0$ and $a_1$. All other coefficients can be determined based upon these two. We therefore write

$$h(\xi) = h_{\text{even}}(\xi) + h_{\text{odd}}(\xi)$$  \hspace{1cm} (7.28)

where

$$h_{\text{even}}(\xi) = a_0 + a_2\xi^2 + a_4\xi^4 + \cdots$$  \hspace{1cm} (7.29)

is the even function of $\xi$, built upon $a_0$, and where

$$h_{\text{odd}}(\xi) = a_1\xi + a_3\xi^3 + a_5\xi^5 + \cdots$$  \hspace{1cm} (7.30)

is the odd function of $\xi$, built upon $a_1$. Thus, we have the solution of the differential equation in terms of two undetermined constants, $a_0$ and $a_1$, which is what we would expect from a second-order differential equation. The final form of the solution, then, is

$$\psi(\xi) = e^{-\frac{1}{2}\xi^2} h(\xi) = e^{-\frac{1}{2}\xi^2} \left\{ \sum_{j=0}^{\infty} a_{2j} \xi^{2j} + \sum_{j=0}^{\infty} a_{2j+1} \xi^{2j+1} \right\}$$  \hspace{1cm} (7.31)

where we use the recursion relation to determine the $a$'s from $a_0$ and $a_1$.

Before we celebrate too much, however, we must determine if this wave function converges as $\xi \to \infty$; otherwise the solution we have obtained is unacceptable. If there is a problem with this function as $\xi$ gets larger it would obviously be due to the larger powers of $\xi$, i.e., larger values of $j$. Let's see what the recursion formula gives us as $j$ gets large. The recursion formula can be written

$$a_{j+2} = \frac{2j + 1 - K}{(j + 1)(j + 2)} a_j$$  \hspace{1cm} (7.32)

or, in the limit of large $j$

$$\lim_{j \to \text{large}} a_{j+2} = \frac{2}{j} a_j$$  \hspace{1cm} (7.33)

or

$$\lim_{j \to \text{large}} \frac{a_{j+2}}{a_j} = \frac{2}{j}$$  \hspace{1cm} (7.34)

If we let $J$ be the value of $j$ where this limit begins to be valid, then we can divide the summation up into two separate pieces; the region below $J$ where the recursion formula
must be used, and the region above \( J \) where the approximation to the recursion formula for large \( j \) can be used. The summation over the even terms, then, becomes

\[
h_{\text{even}}(\xi) = \sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + \sum_{j=J+1}^{\infty} a_{2j} \xi^{2j} \quad (7.35)
\]

or

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} + a_{2J+4} \xi^{2J+4} + a_{2J+6} \xi^{2J+6} + \ldots \quad (7.36)
\]

\[
= \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ 1 + \frac{a_{2J+4} \xi^2}{a_{2J+2}} + \frac{a_{2J+6} \xi^4}{a_{2J+2}} + \ldots \right\}
\]

But for large \( j \) we have

\[
\frac{a_{2J+4}}{a_{2J+2}} = \frac{a_{2(J+2)+2}}{a_{2(J+2)+2}} \approx \frac{2}{(2J + 2)} \\
\frac{a_{2J+6}}{a_{2J+2}} = \frac{a_{2(J+1)+2}}{a_{2(J+1)+2}} \approx \frac{2}{(2J + 4)} \cdot \frac{2}{(2J + 2)}
\]

giving

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ 1 + \frac{\xi^2}{J + 1} + \frac{\xi^4}{(J + 2)(J + 1)} + \ldots \right\} \quad (7.38)
\]

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ \frac{1}{J!} + \frac{\xi^2}{(J+1)J!} + \frac{\xi^4}{(J+2)(J+1)J!} + \ldots \right\}
\]

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ \frac{\xi^{2, J+2}}{J!} + \frac{\xi^{2, J+4}}{(J+1)!} + \frac{\xi^{2, J+6}}{(J+2)!} + \ldots \right\}
\]

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ \frac{\xi^{2, J+1}}{J!} + \frac{\xi^{2, J+3}}{(J+1)!} + \frac{\xi^{2, J+5}}{(J+2)!} + \ldots \right\}
\]

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ \xi^{2, J} \frac{J!}{J!} + \frac{\xi^{2, J+1}}{(J+1)!} + \frac{\xi^{2, J+2}}{(J+2)!} + \ldots \right\}
\]

\[
\sum_{j=0}^{\infty} a_{2j} \xi^{2j} = \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2J+2} \left\{ e^{\xi^2} \right\}
\]
The even solution to the quantum harmonic oscillator problem, therefore, has the form

$$\psi_{\text{even}}(\xi) = e^{-\frac{1}{2} \xi^2} h_{\text{even}}(\xi) = e^{-\frac{1}{2} \xi^2} \left\{ \sum_{j=0}^{J} a_{2j} \xi^{2j} + a_{2J+2} \xi^{2} J! \{ e^{\xi^2} \} \right\}$$

(7.39)

$$\psi_{\text{even}}(\xi) = e^{-\frac{1}{2} \xi^2} h_{\text{even}}(\xi) = e^{-\frac{1}{2} \xi^2} \sum_{j=0}^{J} a_{2j} \xi^{2j} + C \xi^2 e^{\frac{1}{2} \xi^2}$$

which clearly diverges. This same type behavior can be demonstrated for $\psi_{\text{odd}}(\xi)$ as well. So what do we do?

Obviously, our solution is not the infinite series we have derived, because this infinite series diverges. However, if the series were to terminate for some value of $j$, then the function would remain finite. Let's look again at the recursion formula

$$a_{j+2} = \frac{2j + 1 - K}{(j+1)(j+2)} a_{j}$$

(7.40)

If we demand that the term $a_{j+2} = 0$ for $j = \lambda$ (where $a_{j} \neq 0$), we obtain

$$\frac{2\lambda + 1 - K}{(\lambda+1)(\lambda+2)} = 0$$

(7.41)

or

$$2\lambda + 1 - K = 0$$

(7.42)

from which we obtain an equation which fixes the energy of the oscillator

$$K_\lambda = 2E/\hbar \omega = 2\lambda + 1$$

(7.43)

or

$$E_\lambda = \left( \lambda + \frac{1}{2} \right) \hbar \omega \quad \text{where } \lambda = 0, 1, 2, 3, \ldots$$

(7.44)

This gives us a denumerable energy spectrum and implies that the correct solution for our problem is not an infinite series solution for $h(\xi)$, but a polynomial solution for $h(\xi)$, expressed as

$$h_{\lambda}(\xi) = a_{0} + a_{1} \xi + a_{2} \xi^2 + \cdots = \sum_{j=0}^{\lambda} a_{j} \xi^j$$

(7.45)

with different order polynomials being associated with different energies. Now the coefficients satisfy the recursion formula
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\[ a_{j+2} = \frac{2j + 1 - K}{(j + 1)(j + 2)} a_j \]  
(7.46)

where \( K = 2\lambda + 1 \), so the recursion relation depends upon \( \lambda \) as well as \( j \). We express this recursion formula with this substitution to obtain

\[ a_{j+2}^\lambda = \frac{-2(\lambda - j)}{(j + 1)(j + 2)} a_j^\lambda \]  
(7.47)

For \( \lambda = 0 \), (corresponding to \( K_0 = 1 \), which gives \( E_0 = \frac{1}{2} \hbar \omega \)), and we see that the recursion formula

\[ a_{j+2}^0 = \frac{-2(0 - j)}{(j + 1)(j + 2)} a_j^0 \]  
(7.48)

gives zero for all even \( a \)'s other than \( a_0^0 \). But what about the odd values of \( a \)? We require that \( a_1 \) be set equal to zero; otherwise, we would have an infinite series which diverges. Likewise, \( a_0 \) is set equal to zero for the odd polynomials. The polynomials, then can be expressed as:

\[
\begin{align*}
    h_0(\xi) &= a_0^0 \\
    h_1(\xi) &= a_1^1 \xi \\
    h_2(\xi) &= a_2^2 + a_2^2 \xi^2 \\
    h_3(\xi) &= a_3^3 \xi + a_3^3 \xi^3 \\
    h_4(\xi) &= a_4^4 + a_4^4 \xi^2 + a_4^4 \xi^4 \\
    &\vdots
\end{align*}
\]

or, in terms only of the first coefficient in each of the even or odd sequences (note that \( a_0^0 \neq a_0^2 \neq a_0^4 \)):

\[
\begin{align*}
    h_0(\xi) &= a_0^0 \\
    h_1(\xi) &= a_1^1 \xi \\
    h_2(\xi) &= a_2^2 \left\{1 - 2\xi^2\right\} \\
    h_3(\xi) &= a_3^3 \left\{\xi - \frac{2}{3} \xi^3\right\} \\
    h_4(\xi) &= a_4^4 \left\{1 - 4\xi^2 + \frac{4}{3} \xi^4\right\} \\
    &\vdots
\end{align*}
\]

(7.50)

Remember that multiplying the wavefunction solution to Schrödinger's equation by some arbitrary constant does not change the differential equation. This is the reason we are always at liberty to normalize the wave function. The leading coefficient, then, (the one outside the braces) in each of these polynomials will be determined by normalization of the wave function. This means that we are at liberty to multiply or divide the coefficients in each polynomial by some common factor without changing the
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equation in any significant way (we just incorporate that arbitrary change into the arbitrary coefficient). It is customary to choose the constants in the polynomials so that the coefficient of the highest order term of $\xi$ is $(2)^n$. Doing this removes the fractions in the polynomial equations. In addition, we can multiply by a negative so that the sign of the highest order term is always positive. With these changes, the polynomials can be written as

$$
\begin{align*}
  h_0(\xi) &= a_0^0 \\
  h_1(\xi) &= a_1^1 2\xi \\
  h_2(\xi) &= a_0^2 \{ 4\xi^2 - 2 \} \\
  h_3(\xi) &= a_1^3 \{ 8\xi^3 - 12\xi \} \\
  h_4(\xi) &= a_0^4 \{ 16\xi^4 - 48\xi^2 + 12 \} \\
  &\vdots
\end{align*}
$$

The terms inside the braces turn out to be the so-called Hermite polynomials which arise from the Taylor series expansion

$$
e^{-z^2+2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi)$$

and can be generated using the Rodrigues formula

$$H_n(\xi) = (-1)^n e^{\xi^2} \left( \frac{d}{d\xi} \right)^n e^{-\xi^2}$$

**Problem 7.1**
Show that you can obtain the terms in the braces from the Taylor series expansion. Also, prove that the Rodrigues formula works for the first few Hermite polynomials.

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So, apart from the normalization constant, the solutions to the harmonic oscillator problem are just the Hermite polynomials. We next need to determine the normalization constants. The wave functions associated with the various energy states have the form

$$\psi_n(\xi) = h_n(\xi) e^{-\frac{1}{2}\xi^2}$$

The normalization condition for the ground state wave function is

$$\int_{-\infty}^{+\infty} \psi_0^*(x) \psi_0(x) \, dx = \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} \psi_0^*(\xi) \psi_0(\xi) \, d\xi = 1$$

(7.55)
where we recall that \( x = \alpha \xi \), with \( \alpha = \sqrt{\hbar/m\omega} \). Substituting in for the wave function, we have

\[
\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} (a_0^0)^2 e^{-\xi^2} d\xi = 1 \tag{7.56}
\]

\[
\sqrt{\frac{\hbar}{m\omega}} (a_0^0)^2 2 \int_{0}^{+\infty} e^{-\xi^2} d\xi = 1
\]

\[
\sqrt{\frac{\hbar}{m\omega}} (a_0^0)^2 2 \left\{ \frac{1}{2 \pi^2} \right\} = 1
\]

using the Gaussian integral tables. Solving for the coefficient, we have

\[
a_0^0 = \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}} \tag{7.57}
\]

Normalization of the next wave function gives

\[
\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} \psi_2^*(\xi) \psi_2(\xi) d\xi = 1 \tag{7.58}
\]

\[
\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} (a_1^1 2\xi)^2 e^{-\xi^2} d\xi = 1
\]

\[
\sqrt{\frac{\hbar}{m\omega}} (a_1^1)^2 2 \int_{0}^{+\infty} 4\xi^2 e^{-\xi^2} d\xi = 1
\]

\[
\sqrt{\frac{\hbar}{m\omega}} (a_1^1)^2 8 \left\{ \frac{1}{4 \pi^2} \right\} = 1
\]

or, solving for the coefficient,

\[
a_1^1 = \frac{1}{\sqrt{2}} \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}} = \frac{1}{\sqrt{2^1(1!)} \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}}} \tag{7.59}
\]

Continuing this process, we find that the normalization constant can be written

\[
A_n = \frac{1}{\sqrt{2^n(n!)}} \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}} \tag{7.60}
\]

Thus, the solution to the quantum harmonic oscillator problem can be written

\[
\psi_n(\xi) = \frac{1}{\sqrt{2^n(n!)}} \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}} H_n(\xi)e^{-\xi^2/2} \tag{7.61}
\]

or since \( x = \alpha \xi \), with \( \alpha = \sqrt{\hbar/m\omega} \), we write \( \xi = \sqrt{m\omega/\hbar} x \) so that the equation, in terms of \( x \) can be written

\[
\psi_n(x) = \frac{1}{\sqrt{2^n(n!)}} \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{2}} H_n\left( \sqrt{m\omega/\hbar} \ x \right)e^{-m\omega x^2/2\hbar} \tag{7.62}
\]
**Note 1:** Normalization of the second order wave function gives

\[
\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} \psi_2^*(\xi) \psi_2(\xi) \, d\xi = 1
\]

\[
\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{+\infty} \left[ a_0^2 \left\{ 4\xi^2 - 2 \right\} \right]^2 e^{-\xi^2} \, d\xi = 1
\]

\[
\sqrt{\frac{\hbar}{m\omega}} (a_0^2)^2 2 \int_0^{+\infty} \left\{ 16\xi^4 - 16\xi^2 + 4 \right\} e^{-\xi^2} \, d\xi = 1
\]

\[
\sqrt{\frac{\pi\hbar}{m\omega}} (a_0^2)^2 2 \left\{ 16 \cdot \frac{3}{8} \cdot \pi^{\frac{1}{2}} - 16 \cdot \frac{1}{4} \cdot \pi^{\frac{1}{2}} + 4 \cdot \frac{1}{2} \cdot \pi^{\frac{1}{2}} \right\} = 1
\]

\[
\sqrt{\frac{\pi\hbar}{m\omega}} (a_0^2)^2 2 \{ 6 - 4 + 2 \} = 1
\]

\[
\sqrt{\frac{\pi\hbar}{m\omega}} (a_0^2)^2 8 = 1
\]

or, solving for the coefficient,

\[
a_0^2 = \frac{1}{\sqrt{8}} \left[ \frac{m\omega}{\pi\hbar} \right]^\frac{1}{4} = \frac{1}{\sqrt{2^4(2!)}} \left[ \frac{m\omega}{\pi\hbar} \right]^\frac{1}{4} \tag{7.63}
\]

which follows the form stated above.

**Note 2:** Two additional relationships for the Hermite polynomials which often prove useful in calculations are

\[
H_{n+1}(\xi) = 2\xi H_n(\xi) - 2n H_{n-1}(\xi) \tag{7.64}
\]

where one polynomial can be related to the sum of two previously calculated polynomials, and

\[
\frac{\partial H_n(\xi)}{\partial \xi} = 2n H_{n-1}(\xi) \tag{7.65}
\]

where the derivative of a polynomial is related to the previous polynomial.