Chapter Nine
The Quantum Oscillator: Algebraic Method

We now wish to solve the quantum harmonic oscillator problem using quantum mechanical operators and an algebraic technique. As we progress through this formalism, we will implement many of the notational features and the new concepts which we introduced in the last chapter. This solution to the quantum oscillator makes use of the fact that the left-hand-side of the simple harmonic oscillator equation in operator form is composed of the sum of two squared operators

\[
\hat{L} = \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}
\]

which we choose to write in the form

\[
\hat{L} = \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}
\]

The fact that the square of a complex number \(Z\), written \(|Z|^2\) can be written as

\[
|Z|^2 = (x + iy)(x - iy) = x^2 + y^2
\]

sometimes allows us to write the sum of two squares in terms of the square of a single complex number. We want to try this trick here to see if this, indeed, simplifies things. However, we must remember that we are dealing with operators, not just numbers and operators do not generally commute.

Starting with the Hamiltonian in operator form

\[
\hat{H} = \frac{m\omega^2}{2} \left[ \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right]
\]

we will “try” to factor this into two operators

\[
\hat{A}_+ = \sqrt{\frac{m\omega^2}{2}} \left\{ \hat{x} + \frac{i\hat{p}}{m\omega} \right\}
\]

\[
\hat{A}_- = \sqrt{\frac{m\omega^2}{2}} \left\{ \hat{x} - \frac{i\hat{p}}{m\omega} \right\}
\]

Multiplying these operators together gives

\[
\hat{A}_+ \hat{A}_- = \frac{m\omega^2}{2} \left\{ \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega} [\hat{x}, \hat{p}] \right\}
\]
which almost works (except for the $xp$ commutator). The Hamiltonian for the quantum oscillator can be expressed in terms of the $\hat{A}_+$ and $\hat{A}_-$ operators by the equation

$$\hat{H} = \frac{m\omega^2}{2} \left[ \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right] = \hat{A}_+\hat{A}_- + \frac{i\omega}{2}[\hat{x},\hat{p}] = \hat{A}_+\hat{A}_- - \frac{\hbar \omega}{2}$$

(9.7)

It seems that all we have accomplished is to rewrite the Schrödinger equation in terms of two new operators $\hat{A}_+$ and $\hat{A}_-$ defined by the equations

$$\hat{A}_\pm = \sqrt{\frac{m\omega^2}{2}} \left\{ \hat{x} \pm i \frac{\hat{p}}{m\omega} \right\}$$

(9.8)

But these operators can be utilized to solve the quantum oscillator problem in a fairly straightforward manner. Obviously, the Hamiltonian has units of energy, and thus the product $\hat{A}_+\hat{A}_-$ also have units of energy. It is customary to redefine these operators so that they are dimensionless. To make the operators dimensionless, we introduce two new operators $\hat{a}_+$ and $\hat{a}_-$ such that

$$\hat{H} = \hat{A}_+\hat{A}_- - \frac{\hbar \omega}{2} = \left( \frac{\hat{A}_+\hat{A}_-}{\hbar \omega} - \frac{1}{2} \right) \hbar \omega = \left( \hat{a}_+\hat{a}_- - \frac{1}{2} \right) \hbar \omega$$

(9.9)

where

$$\frac{\hat{A}_+\hat{A}_-}{\hbar \omega} = \hat{a}_+\hat{a}_- \Rightarrow \hat{a}_\pm = \frac{\hat{A}_\pm}{\sqrt{\hbar \omega}}$$

(9.10)

Thus, using our dimensionless operators given by

$$\hat{a}_\pm = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} \pm i \frac{\hat{p}}{m\omega} \right\}$$

(9.11)

the Hamiltonian for the quantum oscillator can now be expressed by the equation

$$\hat{H} = \left( \hat{a}_+\hat{a}_- - \frac{1}{2} \right) \hbar \omega$$

(9.12)

We now want to examine the nature of these two operators. First, we might inquire if the two operators are Hermitian? Since these operators, as defined, are a summation of Hermitian operators $\hat{x}$ and $\hat{p}$, you might be tempted to think that the $\hat{a}_+$ and $\hat{a}_-$ operators are also Hermitian (or self-adjoint). But remember that the sum of Hermitian operators multiplied by arbitrary constants is Hermitian only if the constants are all real. That is not the case for these operators! Thus, neither $\hat{a}_+$ nor $\hat{a}_-$ are self-adjoint – in fact, you should be able to convince yourself that the adjoint of $\hat{a}_+$ is just $\hat{a}_-$ (and vice versa)! For this reason, it is conventional to express our operators as simply $\hat{a}$ and $\hat{a}^\dagger$, as follows.
\[
\hat{a} \leftrightarrow \hat{a}_+ = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} + i \frac{\hat{p}}{m\omega} \right\} \\
\hat{a}^\dagger \leftrightarrow \hat{a}_- = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} - i \frac{\hat{p}}{m\omega} \right\}
\]

(9.13)

Since neither of the operators \(\hat{a}\) or \(\hat{a}^\dagger\) are self-adjoint, neither operator corresponds to the measurement of a physical quantity. The combination of these two operators, \(\hat{a}\hat{a}^\dagger\), however, is Hermitian: this combination is equal to the Hermitian operator \(\hat{H}\) plus a real constant

\[
\hat{H} = \left( \hat{a}\hat{a}^\dagger - \frac{1}{2} \right) \hbar \omega
\]

(9.14)

This means that the combination \(\hat{a}\hat{a}^\dagger\) must correspond to some measurable, dimensionless quantity (some number) corresponding to a given state of the quantum oscillator.

The commutation relationship between the two operators \(\hat{a}\) and \(\hat{a}^\dagger\) can easily be determined from the definition of the operators and shown to be

\[
\left[ \hat{a}, \hat{a}^\dagger \right] = 1
\]

(9.15)

and, therefore, that

\[
\left[ \hat{a}^\dagger, \hat{a} \right] = -1
\]

(9.16)

**Problem 9.1**

Starting from the definition of the operators \(\hat{a}\) and \(\hat{a}^\dagger\), show that

\[
\left[ \hat{a}, \hat{a}^\dagger \right] = 1
\]

Notice that this commutator relationship implies that

\[
\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1
\]

(9.17)

\[
\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1
\]
Utilizing the commutator relationships, the Hamiltonian operator can be written in two different, equivalent ways

\[
\hat{H} = \hbar \omega \left( \hat{a} \hat{a}^\dagger - \frac{1}{2} \right) \\
\hat{H} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)
\]

(9.18)

It should be obvious from this last equation that the Hamiltonian and the combination operator \( \hat{a}^\dagger \hat{a} \) have common eigenvectors. The combination operator \( \hat{a}^\dagger \hat{a} \) is a dimensionless, Hermitian operator, with real eigenvalues. These eigenvalues are numbers related to the energy allowed for each eigenvector, and so we define this combination operator as the number operator

\[
\hat{N} = \hat{a}^\dagger \hat{a}
\]

(9.19)

Thus, the eigenvectors which will describe the different allowed states of the quantum harmonic oscillator are also the eigenvectors associated with this Hermitian number operator \( \hat{N} \) and the Hamiltonian can be written as

\[
\hat{H} = \hbar \omega \left( \hat{N} + \frac{1}{2} \right)
\]

(9.20)

We will develop the equations for these eigenvectors and their corresponding eigenvalues in what follows.

We begin by writing the eigenvalue equation

\[
\hat{N} |n\rangle = n |n\rangle \\
\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle
\]

(9.21)

where \( |n\rangle \) represents an arbitrary eigenvector of the number operator (and the Hamiltonian), and \( n \) represents the real eigenvalue associated with that eigenvector. We now want to determine the effect of the two operators \( \hat{a}^\dagger \) and \( \hat{a} \) on the eigenvectors of the number operator. When \( \hat{a}^\dagger \) operates on an eigenvector of \( \hat{N} \), it produces a new vector that we will express in the following way

\[
\hat{a}^\dagger |n\rangle = |\hat{a}^\dagger n\rangle
\]

(9.22)

Now, if we operate on this new vector with the number operator

\[
\hat{N} |\hat{a}^\dagger n\rangle = \hat{a}^\dagger \hat{a} \hat{a} |n\rangle
\]

(9.23)

and make use of the commutator relationships, we find that

\[
\hat{N} |\hat{a}^\dagger n\rangle = \hat{a}^\dagger \left( \hat{a}^\dagger \hat{a} + 1 \right) |n\rangle = \hat{a}^\dagger (n + 1) |n\rangle = (n + 1) |\hat{a}^\dagger n\rangle
\]

(9.24)
So we see that the operator $\hat{a}^\dagger$ acting on an eigenvector $|n\rangle$ whose eigenvalue is $n$ produces a new eigenvector or $\hat{N}$ with an eigenvalue of $n + 1$, or

$$\hat{a}^\dagger |n\rangle = C_n^+ |n + 1\rangle \quad (9.25)$$

The constant $C_n^+$ is introduced in this last equation because we assume that the eigenvectors of the operator $\hat{N}$ are normalized, but we are not assured that the action of $\hat{a}^\dagger$ on a normalized eigenvector produces a normalized eigenvector.

Similarly, if we operate on an eigenvector of $\hat{N}$ with the $\hat{a}$ operator, we obtain a new vector

$$\hat{a} |n\rangle = |\hat{a}n\rangle \quad (9.26)$$

Operating on this new vector with the number operator, we obtain

$$\hat{N} |\hat{a}n\rangle = \left(\hat{a}\hat{a}^\dagger\right) |\hat{a}n\rangle = \left(\hat{a}\hat{a}^\dagger - 1\right) |\hat{a}n\rangle = \hat{a} |\hat{a}n\rangle - |\hat{a}n\rangle = (n - 1) |\hat{a}n\rangle \quad (9.27)$$

So we see that the operator $\hat{a}$ acting on an eigenvector $|n\rangle$ whose eigenvalue is $n$ produces a new eigenvector of $\hat{N}$ with an eigenvalue of $n - 1$, or

$$\hat{a} |n\rangle = C_n^- |n - 1\rangle \quad (9.28)$$

Thus the $\hat{a}^\dagger$ operator operating on an eigenvector of the number operator produces a new eigenvector with an eigenvalue increased by one unit, while the action of the $\hat{a}$ operator operating on an eigenvector produces a new eigenvector with an eigenvalue decreased by one unit. For this reason, these two operators together are often called “ladder” operators: $\hat{a}^\dagger$ being the “step-up” operator, and $\hat{a}$ being the “step-down” operator. Subsequent operation of the $\hat{a}^\dagger$ or $\hat{a}$ operators on a given eigenstate of the number operator will move up and down through a series of allowed eigenvectors with corresponding eigenvalues separated by unity. Notice that we do not yet know the actual eigenvalues, nor the functional form (either in position or momentum representation) of the eigenfunctions. This exercise simply allows us to generate one eigenvector from another whose eigenvalues differ by unity.

To determine the allowed eigenvalues, we will first consider the state designated by $|\hat{a}n\rangle$. Recall that the normalization condition for an arbitrary state vector is given by

$$\langle \psi | \psi \rangle = \int \langle \psi | x \rangle \langle x | \psi \rangle \, dx = \int |\psi^*(x)|^2 \, dx = \int |\psi(x)|^2 \, dx = 1 \quad (9.29)$$

The fact that this quantity is related to the absolute magnitude squared, indicates that the “norm” of a vector can never be less than zero. Also, our interpretation of $\langle \psi | \psi \rangle$ as the projection of the vector $|\psi\rangle$ onto itself would indicate that this quantity must be positive.
definite. Thus, in general, the norm of any vector must satisfy the equation
\[
\langle \psi | \psi \rangle \geq 0
\]  
(9.30)
This must also be true of the vector \( | \hat{a} \rangle \langle n | \)
\[
\langle \hat{a} \rangle \langle n | \hat{a} \rangle \rangle \geq 0
\]  
(9.31)
But the adjoint of \( \hat{a} \) is \( \hat{a}^\dagger \), so we can write this last equation as
\[
\langle n | \hat{a}^\dagger \hat{a} | n \rangle = n \geq 0
\]  
(9.32)
which indicates that all of the eigenvalues of the quantum oscillator must be positive
definite. [Note: If you examine the norm of the vector \( | \hat{a}^\dagger \rangle \), you will find that you
obtain the relationship \( n + 1 \geq 0 \), \( \Rightarrow n \geq -1 \)! But because we demand that both of
these vectors have a positive definite norm, we are forced to take the all inclusive
condition that \( n \geq 0 \).
Since there is a known lower limit for our eigenvalues, we know that there is an
eigenvector \( | n_{\text{min}} \rangle \) such that
\[
\hat{N} | n_{\text{min}} \rangle = n_{\text{min}} | n_{\text{min}} \rangle
\]  
(9.33)
\[
\hat{a}^\dagger \hat{a} | n_{\text{min}} \rangle = n_{\text{min}} | n_{\text{min}} \rangle
\]  
\[
\hat{a}^\dagger (\hat{a} | n_{\text{min}} \rangle) = n_{\text{min}} | n_{\text{min}} \rangle
\]
The term in parentheses in this last expression, however, must be equivalent to zero,
since the step-down operator cannot produce a state with a lower eigenvalue the \( n_{\text{min}} \)!
This would imply that the minimum eigenvalue is zero, and that the “ground-state”
eigenvector is designated as \( | 0 \rangle \). Thus, we have for the ground state.
\[
\hat{N} | 0 \rangle = 0 | 0 \rangle
\]  
(9.34)
All eigenvectors associated with higher value eigenvalues can now be generated by
operating on the ground state eigenvector with the step-up operator \( \hat{a}^\dagger \)
\[
\hat{a}^\dagger | 0 \rangle = C_0^+ | 1 \rangle , \text{ etc.}
\]  
(9.35)
Thus, all the eigenvalues of the number operator must be integers! This means that the
Hamiltonian operator operating on an eigenvector of the quantum oscillator must give
\[
\hat{H} | n \rangle = \hbar \omega \left( \hat{N} + \frac{1}{2} \right) | n \rangle = \hbar \omega \left( n + \frac{1}{2} \right) | n \rangle
\]  
(9.36)
where \( n \) is an integer, and that the ground state energy is given by \( \hbar \omega /2 \).
To generate all the eigenvectors, we need to find the values of the constants \( C_n^+ \)
and \( C_n^- \). First consider the quantity
\[
\langle n | \hat{a}^\dagger | n - 1 \rangle = C_{n-1}^+ \langle n | n \rangle = C_{n-1}^+ \]  
(9.37)
This same expression can also be evaluated as follows:

\[ \langle n | \hat{a}^\dagger | n - 1 \rangle = \langle \hat{a} | n | n - 1 \rangle = (n - 1 | \hat{a} | n \rangle^* = (C_n^-)^* \]  

(9.38)

which means that

\[ C_{n-1}^+ = (C_n^-)^* \]  

(9.39)

Now, calculating the expectation value of the number operator, we find that

\[ n = \langle n | \hat{a}^\dagger \hat{a} | n \rangle = C_n^- \langle n | \hat{a}^\dagger | n - 1 \rangle = C_n^- C_{n-1}^+ \langle n | n \rangle = C_n^- C_{n-1}^+ \]  

(9.40)

These last two equations taken together, give

\[ n = |C_n^-|^2 \Rightarrow C_n^- = \sqrt{n} \]  

(9.41)

where we have assumed that the coefficients are real. Looking back at the equation

\[ C_{n-1}^+ = (C_n^-)^* \]  

(9.42)

and making a shift in the dummy variable \((n \rightarrow n + 1)\) we obtain

\[ C_n^+ = (C_{n+1}^-)^* = \sqrt{n + 1} \]  

(9.43)

[Note: The proof that the coefficients \(C_n^+\) and \(C_n^-\) are real is non-trial, and will not be presented here.] Thus, we can now write the action of our step-up and step-down operators as

\[
\begin{align*}
\hat{a}^\dagger | n \rangle &= \sqrt{n+1} | n + 1 \rangle \\
\hat{a} | n \rangle &= \sqrt{n} | n - 1 \rangle
\end{align*}
\]  

(9.44)

Before proceeding to find the correct functional form for our eigenfunctions, let's check to see if our development is consistent. We have concluded that the eigenvalues are all integers and are separated by unity, and that the ground-state eigenvalue is zero. This means that we should be able to generate all the eigenvectors starting with the ground state eigenvector \(|0\rangle\)

\[
\begin{align*}
\hat{a}^\dagger |0\rangle &= \sqrt{0+1} |1\rangle = \sqrt{1} |1\rangle \\
\hat{a}^\dagger |1\rangle &= \sqrt{1+1} |2\rangle = \sqrt{2} |2\rangle \\
\hat{a}^\dagger |2\rangle &= \sqrt{2+1} |3\rangle = \sqrt{3} |3\rangle \\
&\vdots
\end{align*}
\]  

(9.45)
The third eigenvector, then, can be generated as follows:

\[
|3\rangle = \frac{1}{\sqrt{3}} \hat{a}^\dagger |2\rangle
\]

\[
= \frac{1}{\sqrt{3}} \hat{a}^\dagger \frac{1}{\sqrt{2}} \hat{a}^\dagger |1\rangle
\]

\[
= \frac{1}{\sqrt{3}} \hat{a}^\dagger \frac{1}{\sqrt{2}} \hat{a}^\dagger \frac{1}{\sqrt{1}} |0\rangle
\]

\[
= \frac{1}{\sqrt{3} \cdot 2 \cdot 1} (\hat{a}^\dagger)^3 |0\rangle
\]

of, in general,

\[
|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle
\]

Now let's check going the other way. We will start at the eigenvector \(|3\rangle\) and work our way down

\[
\hat{a} |3\rangle = \sqrt{3} |2\rangle
\]

\[
\hat{a} |2\rangle = \sqrt{2} |1\rangle
\]

\[
\hat{a} |1\rangle = \sqrt{1} |0\rangle
\]

\[
\hat{a} |0\rangle = \sqrt{0} |-1\rangle = |\emptyset\rangle
\]

The eigenvector \(|-1\rangle\) is not strictly allowed – it is inserted here simply to follow the pattern. Since the step-down operator operating on the ground state gives an eigenvalue of zero, we define the vector produced by that operation as the null vector \(|\emptyset\rangle\).

Notice what happens if we assume that the eigenvalues are not integers; for example assume that one such eigenvector is given by \(|3/2\rangle\).

\[
\hat{a} |3/2\rangle = \sqrt{3/2} |1/2\rangle
\]

\[
\hat{a} |1/2\rangle = \sqrt{1/2} |-1/2\rangle
\]

This last step leads to a contradiction, since there is no state with eigenvalue less than zero.

We are now in a position to derive the functional form of the eigenvectors. We begin by determining the position-dependent functional form of the ground-state wave function. Recall that the step-up and step-down operators are given by

\[
\hat{a} \leftrightarrow \hat{a}_+ = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} + i \frac{\hat{p}}{m\omega} \right\}
\]

\[
\hat{a}^\dagger \leftrightarrow \hat{a}_- = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} - i \frac{\hat{p}}{m\omega} \right\}
\]
Operating on the ground state vector with the step-down operator, we obtain
\[
\widehat{a}|0\rangle = 0 = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} + i \frac{\hat{p}}{m\omega} \right\}|0\rangle
\] (9.51)

To obtain the equivalent operator in position representation we simply change each of the operators \(\hat{\rho}\) and \(\hat{\sigma}\) to their corresponding position representation, and write the position representation of the ground state wave function as \(\langle x|0\rangle\), we have the differential equation
\[
\sqrt{\frac{m\omega}{2\hbar}} \left\{ x + \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right\} \langle x|0\rangle = 0
\] (9.52)

which we write as
\[
\frac{\partial}{\partial x} \langle x|0\rangle = -\frac{m\omega}{\hbar} x \langle x|0\rangle
\] (9.53)

This differential equation is easily solvable to obtain
\[
\langle x|0\rangle = A_0 e^{-m\omega x^2/2\hbar}
\] (9.54)

where \(A_0\) is the normalization constant for the ground state. It is left as an exercise for the student to show that the normalization constant is given by
\[
A_0 = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4}
\] (9.55)

**Problem 9.2**
Show that the normalization constant of the ground state wavefunction for the quantum harmonic oscillator is given by
\[
A_0 = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4}
\]

We therefore have the fundamental building blocks upon which we can construct an entire set of eigenfunctions and eigenvalues. The functional form of the ground state wave function, and the ground state energy are given by:
\[
\langle x|0\rangle = \langle x|E_0\rangle = A_0 e^{-m\omega x^2/2\hbar} \quad \quad E_0 = \hbar \omega / 2
\] (9.56)

Additional wavefunctions can be generated by operating on the ground state wave function with the step-up operator expressed in position representation
\[
\langle x|n\rangle = \frac{1}{\sqrt{n!}} \left( \widehat{a}^\dagger \right)^n \langle x|0\rangle
\] (9.57)
where

\[
(\hat{a}_x^\dagger)^n = \left(\sqrt{\frac{m\omega}{2\hbar}}\right)^n \left\{ x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right\}^n = \left(\sqrt{\frac{\beta}{2}}\right)^n \left\{ x - \frac{1}{\beta} \frac{\partial}{\partial x} \right\}^n
\]  

(9.58)

where \( \beta = \frac{m\omega}{\hbar} \). The first few eigenfunctions for our quantum harmonic oscillator, then, are

\[
\langle x | 0 \rangle = \langle x | E_0 \rangle = \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} 
\]

\[
\langle x | 1 \rangle = \frac{1}{\sqrt{1!}} \left(\sqrt{\frac{\beta}{2}}\right)^1 \left\{ x - \frac{1}{\beta} \frac{\partial}{\partial x} \right\}^1 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} \right]
\]

\[
\langle x | 2 \rangle = \frac{1}{\sqrt{2!}} \left(\sqrt{\frac{\beta}{2}}\right)^2 \left\{ x - \frac{1}{\beta} \frac{\partial}{\partial x} \right\}^2 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} \right]
\]

\[
\langle x | 3 \rangle = \frac{1}{\sqrt{3!}} \left(\sqrt{\frac{\beta}{2}}\right)^3 \left\{ x - \frac{1}{\beta} \frac{\partial}{\partial x} \right\}^3 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} \right]
\]

Letting \( y^2 = \beta x^2 \) to simplify these relationships

\[
\beta x^2 = y^2 \Rightarrow x = \frac{1}{\sqrt{\beta}} y 
\]

\[
\frac{\partial}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = \sqrt{\beta} \frac{\partial}{\partial y}
\]

we obtain

\[
\langle x | 0 \rangle = \langle x | E_0 \rangle = \left(\frac{\beta}{\pi}\right)^{1/4} e^{-y^2/2} 
\]

\[
\langle x | 1 \rangle = \frac{1}{\sqrt{1!}} \left(\frac{1}{\sqrt{2}}\right)^1 \left\{ y - \frac{\partial}{\partial y} \right\}^1 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-y^2/2} \right]
\]

\[
\langle x | 2 \rangle = \frac{1}{\sqrt{2!}} \left(\frac{1}{\sqrt{2}}\right)^2 \left\{ y - \frac{\partial}{\partial y} \right\}^2 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-y^2/2} \right]
\]

\[
\langle x | 3 \rangle = \frac{1}{\sqrt{3!}} \left(\frac{1}{\sqrt{2}}\right)^3 \left\{ y - \frac{\partial}{\partial y} \right\}^3 \left[ \left(\frac{\beta}{\pi}\right)^{1/4} e^{-y^2/2} \right]
\]

or
\begin{align*}
\langle x|0 \rangle &= \langle x|E_0 \rangle = \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \\
\langle x|1 \rangle &= \frac{1}{\sqrt{1!}} \left( \frac{1}{\sqrt{2}} \right)^1 2y \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \\
\langle x|2 \rangle &= \frac{1}{\sqrt{2!}} \left( \frac{1}{\sqrt{2}} \right)^2 \left\{ y - \frac{\partial}{\partial y} \right\} 2y \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \\
&= \frac{1}{\sqrt{2!}} \left( \frac{1}{\sqrt{2}} \right)^2 (4y^2 - 2) \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \\
\langle x|3 \rangle &= \frac{1}{\sqrt{3!}} \left( \frac{1}{\sqrt{2}} \right)^3 \left\{ y - \frac{\partial}{\partial y} \right\} (4y^2 - 2) \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \\
&= \frac{1}{\sqrt{3!}} \left( \frac{1}{\sqrt{2}} \right)^3 (8y^3 - 12y) \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \\
&\ldots
\end{align*}

Thus, the solution for the position wave functions can be expressed by the equation

\begin{equation}
\langle x|n \rangle = \frac{1}{\sqrt{n!}} \left( \frac{1}{\sqrt{2}} \right)^n \ H_n(y) \left[ \left( \frac{\beta}{\pi} \right)^{1/4} e^{-y^2/2} \right] \tag{9.63}
\end{equation}

where $H_n(y)$ are the Hermite polynomials; the first five of which are listed below.

<table>
<thead>
<tr>
<th>Hermite Polynomials $H_n(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
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<tr>
<td>-----</td>
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<tr>
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<td>3</td>
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<tr>
<td>4</td>
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<td>5</td>
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Now, we wish to examine the expectation values of the position and the momentum operators in the various energy eigenstates of the quantum oscillator problem. Recalling
the definition of the step-up and step-down operators

\[ \hat{a} \leftrightarrow \hat{a}_+ = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} + i \frac{\hat{p}}{m\omega} \right\} \]  
\[ \hat{a}^\dagger \leftrightarrow \hat{a}_- = \sqrt{\frac{m\omega}{2\hbar}} \left\{ \hat{x} - i \frac{\hat{p}}{m\omega} \right\} \]  

(9.64)

we can solve for the position and momentum operators in terms of these step-up and step-down operators to obtain

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^\dagger + \hat{a}) \]  
\[ \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}) \]  

(9.65)

**Problem 9.3**

Derive the expressions for the position and momentum operators for the quantum oscillator in terms of the step-up and step-down operators.

The “matrix elements” of the position and momentum operators, then are given by

\[ \langle n | \hat{x} | k \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left( \langle n | \hat{a}^\dagger | k \rangle + \langle n | \hat{a} | k \rangle \right) \]  
\[ = \sqrt{\frac{\hbar}{2m\omega}} \left( \sqrt{k + 1} \delta_{n,k+1} + \sqrt{k} \delta_{n,k-1} \right) \]  

(9.66)

and

\[ \langle n | \hat{p} | k \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \left( \langle n | \hat{a}^\dagger | k \rangle - \langle n | \hat{a} | k \rangle \right) \]  
\[ = i\sqrt{\frac{m\omega\hbar}{2}} \left( \sqrt{k + 1} \delta_{n,k+1} - \sqrt{k} \delta_{n,k-1} \right) \]  

(9.67)

**Problem 9.4**

Determine the expectation value of the position and the momentum in the ground state of the quantum oscillator. Also determine the expectation value of the square of the position and the momentum in the ground state. Determine the numerical value of $\Delta x \Delta p$ for the ground state. What can you say about these same quantities for other energy states?
Problem 9.5
Assume that the general wave function for a quantum oscillator at time $t = 0$ is one in which there is equal probability of measuring either of the two lowest energy values. Determine the expectation value of $x$ and $p$ for this general wave function. Are these two expectation values time dependent?

Problem 9.6
By following the techniques used in the notes, determine the functional form in momentum representation of the ground state of the quantum oscillator. Also generate the momentum representation of the first and second excited state wave functions.