The Development of Lagrangian Mechanics

**Newton's Relations and D'Alembert's Equation**

Consider a particle of mass \( m \) which is moving through three-dimensional space subject to some combination of external forces \( \vec{F} \). This particle will follow some specific path through space which is controlled by the applied forces. These forces may be due to external gravitational or electric fields and/or certain constraint forces, such as the force due to a string or surface of contact. Newton's second law, as applied to this particle, can be expressed by the vector equation

\[
\vec{F} = m \frac{d^2\vec{r}}{dt^2} \tag{1.1}
\]

where \( \vec{F} \) is understood to be the vector sum of all external forces acting on the particle. This last equation is equivalent to the three scalar equations

\[
F_x = m \ddot{x} \quad F_y = m \ddot{y} \quad F_z = m \ddot{z} \tag{1.2}
\]

where \( x, y, \) and \( z \) are the coordinates of the three-dimensional vector \( \vec{r} \) which locates the particle in space. These coordinates are expressed in terms of a right-handed, *inertial* coordinate system.

The work done on this particle by the applied forces as the particle moves from one point to another along its path can be expressed by the equation

\[
dW = F_x \, dx + F_y \, dy + F_z \, dz \tag{1.3}
\]

where the symbols \( dx, dy, \) and \( dz \) represent *differential displacements* along the actual path followed by the particle. The differential length \( ds \) along this path can be determined from the equation

\[
ds^2 = dx^2 + dy^2 + dz^2 \tag{1.4}
\]

Now, using equations (1.2) and (1.3), we have

\[
dW = m \ddot{x} \, dx + m \ddot{y} \, dy + m \ddot{z} \, dz = F_x \, dx + F_y \, dy + F_z \, dz \tag{1.5}
\]

where the right hand side of this last equation is the net work done by all the applied external forces during an infinitesimal displacement, and the left hand side is the corresponding change in the kinetic energy of the particle during that same displacement. This last equation is known as “D'Alembert's equation”.

**Virtual Displacements and Virtual Work**

In what follows, we will introduce the concept of *virtual* displacements and *virtual* work. In order to make these concepts concrete, we will apply them to a simple example. Consider a mass \( m \) moving along the surface of a spherical bowl, as shown in the diagram on the next page. The particle moves under the influence of an external gravitational field, but is constrained to move along the inner surface of the spherical bowl. The magnitude of this constraint force is not known *a priori*, although the
direction is known. It is a normal or contact force which act perpendicular to the point of contact.

The motion of this particle along the surface of the bowl can, of course, be described in terms of the components $x$, $y$, and $z$, but this is not the best set of components to use in this particular case. It is obvious that the $x$, $y$, and $z$ components of the mass are not independent as the mass moves along the surface of the bowl - i.e., a change in $x$ must be associated with a corresponding change in $y$ and/or $z$ because of the constraint. For this particular problem, a better set of components would be the angles $\theta$ and $\phi$, and the radius of the bowl $R$ because the particle is constrained to move along the curve of the bowl where $R = R_0 = constant$! This last equation is known as an equation of constraint, since it indicates the fact that the particle is constrained to move along the spherical surface of the bowl. Thus, although the general motion of a particle in three dimensions requires three independent variables $(x, y, z)$ or $(R, \theta, \phi)$, the equation of constraint restricts the value of $R$ to a particular value $R_0$, thus reducing the number of independent variables by one (the number of constraint equations).

**Difficulties Imposed by Constraints** In general, the presence of constraints introduce two types of difficulties when solving problems in mechanics. First, as illustrated in the example above, the coordinates are no longer independent, but are related by the equations of constraint. In some cases this may actually simplify the
problem, provided appropriate coordinates are used to describe the motion of the system. The second difficulty comes from the fact that the constraint forces are not know a priori. They are among the unknowns of the problem. All we usually know is the effect of these constraints on the motion of the particle.

**Generalized Coordinates** The first of these two problems is addressed by introducing a set of generalized coordinates. These are a set of coordinates which can completely describe the motion of the system, but which are independent and consistent with constraints. Thus, there are as many generalized coordinates as degrees of freedom in the system. To illustrate the use of generalized coordinates, we again look at our example problem.

As mentioned above, the \((R, \theta, \phi)\) coordinates are a better choice for our problem than the \((x, y, z)\) coordinates. This is because the \((R, \theta, \phi)\) coordinates consistent with the constraints of the system, when we require that \(R = R_0\). Once this is done, we realize that the \(\theta\) and \(\phi\) motion are totally independent of each other in this particular system. In general, the relationship between the \((x, y, z)\) coordinates and the \((R, \theta, \phi)\) coordinates can be expressed in terms of the transformation equations

\[
\begin{align*}
x &= R \sin \theta \cos \phi \\
y &= R \sin \theta \sin \phi \\
z &= R [1 - \cos \theta]
\end{align*}
\]  

(1.6)
The additional equation of constraint, \(R = R_0\), can be used to reduce the number of independent variables from three to two.

These transformation equations are often written in a more formal manner

\[
\begin{align*}
x &= x(q_1, q_2, q_3) \\
y &= y(q_1, q_2, q_3) \\
z &= z(q_1, q_2, q_3)
\end{align*}
\]  

(1.7)
or

\[
\begin{align*}
x_i &= x_i(q_1, q_2, q_3)
\end{align*}
\]  

(1.8)
where \(x_i\) \((i = 1, 2, 3)\) represents all three of the rectangular components of the location vector, and where \(q_i\) \((i = 1, 2, 3)\) represents some other (usually more convenient) set of coordinates which may be used to locate the particle.

We can determine how changes in one set of coordinates will manifest themselves in the other coordinates by taking the partial derivatives

\[
\begin{align*}
dx &= \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 \\
dy &= \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3 \\
dz &= \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3
\end{align*}
\]  

(1.9)
For the particular problem we are examining, where \( q_1 = R, q_2 = \theta \), and \( q_3 = \phi \), these equations take on the form

\[
\begin{align*}
\dot{x} &= \sin \theta \cos \phi \, dR + R \cos \theta \cos \phi \, d\theta - R \sin \theta \sin \phi \, d\phi \\
\dot{y} &= \sin \theta \sin \phi \, dR + R \cos \theta \sin \phi \, d\theta + R \sin \theta \cos \phi \, d\phi \\
\dot{z} &= (1 - \cos \theta) dR + R \sin \theta \, d\theta
\end{align*}
\] (1.10)

If we apply the equation of constraint, where \( R = R_0 \Rightarrow dR = 0 \), these equations reduce to

\[
\begin{align*}
\dot{x} &= R_0 \cos \theta \cos \phi \, d\theta - R_0 \sin \theta \sin \phi \, d\phi \\
\dot{y} &= R_0 \cos \theta \sin \phi \, d\theta + R_0 \sin \theta \cos \phi \, d\phi \\
\dot{z} &= R_0 \sin \theta \, d\theta
\end{align*}
\] (1.11)

and the generalized coordinates for this particular problem are the coordinates \( \theta \) and \( \phi \).

**Time Dependent Coordinate Systems** Although the specific example we are considering is not one in which the coordinates are changing in time, we will often find it convenient to work with moving coordinates. In those cases, the components are functions of time, and we must include this in our transformation equations. Thus, the general transformation equation between the \((x, y, z)\) coordinates and the \((R, \theta, \phi)\) coordinates should be expressed as

\[
\begin{align*}
\dot{x} &= \frac{\partial x}{\partial q_1} \dot{q}_1 + \frac{\partial x}{\partial q_2} \dot{q}_2 + \frac{\partial x}{\partial q_3} \dot{q}_3 + \frac{\partial x}{\partial t} \dot{t} \\
\dot{y} &= \frac{\partial y}{\partial q_1} \dot{q}_1 + \frac{\partial y}{\partial q_2} \dot{q}_2 + \frac{\partial y}{\partial q_3} \dot{q}_3 + \frac{\partial y}{\partial t} \dot{t} \\
\dot{z} &= \frac{\partial z}{\partial q_1} \dot{q}_1 + \frac{\partial z}{\partial q_2} \dot{q}_2 + \frac{\partial z}{\partial q_3} \dot{q}_3 + \frac{\partial z}{\partial t} \dot{t}
\end{align*}
\] (1.12)

**Elimination of Forces of Constraint** We now want to consider three types of displacement. First we consider an arbitrary displacement which may actually violate constraint forces. (This would mean, for example, that a small change in \( R \), which we will call \( \delta R \) might occur which would actually slightly deform the spherical bowl.) This displacement is called an arbitrary virtual displacement. The work done during such an arbitrary virtual displacement clearly depends upon all applied forces, including the forces of constraint. We will reserve the \( \delta \) notation for virtual displacements.

The second type of displacement which we want to consider is one in which the displacement \( \delta s \) is in conformity with the constraints, but which takes place during a time interval \( \delta t \) in which moving constraints or coordinate systems may change their positions. This type of motion is called a constrained virtual displacement. Again, the work done during such an arbitrary virtual displacement still depends upon all applied external forces, including the forces of constraint. The validity of this last statement can be illustrated by considering the diagram below. During the time interval \( \delta t \), the bead moves a distance \( \delta s \) while the rigid rod moves through an angle \( \delta \phi \). As can be seen in the diagram, the displacement \( \delta s \) during that time interval has a component both along the rigid rod and perpendicular to it. Thus there is some work done by the rod on the bead.
The third and final type of displacement we want to consider is one in which any displacement $\delta s$ is \textit{in conformity with the constraints of the system, but holding time fixed}. We will call this type of displacement an \textit{instantaneous constrained virtual displacement}. In this type of virtual displacement, all moving components are fixed in time and all motion is considered to be consistent with the constrains. For such virtual displacements, the work done by the constraint forces \textit{must be zero}. It is often this final type of displacement which is refered to when one encounters the term “virtual displacement”.

\textbf{Lagrange's Equation}

To understand the significance of virtual displacements, look back at D'Alembert's equation. If we consider only instantaneous, constrained virtual displacements in this equation, we write the virtual work as

$$
\delta W = m\dot{x}\delta x + m\dot{y}\delta y + m\dot{z}\delta z = F_x \delta x + F_y \delta y + F_z \delta z
$$

(1.13)

where all forces of constraint may be ignored! This means that the only forces which we include on the right hand side of this last equation are forces due to gravity, electrostatics, or other known forces, but not forces of constraint arising from walls or strings, etc.

We want to make this explicit by writing out the \textit{virtual displacements} $\delta x$, $\delta y$, and $\delta z$ in terms of the \textit{generalized coordinates} (the $q_i$'s) \textit{which are independent
coordinates consistent with system constraints

\[
\delta x = \frac{\partial x}{\partial q_1} \delta q_1 + \frac{\partial x}{\partial q_2} \delta q_2 + \frac{\partial x}{\partial q_3} \delta q_3
\]

\[
\delta y = \frac{\partial y}{\partial q_1} \delta q_1 + \frac{\partial y}{\partial q_2} \delta q_2 + \frac{\partial y}{\partial q_3} \delta q_3
\]

\[
\delta z = \frac{\partial z}{\partial q_1} \delta q_1 + \frac{\partial z}{\partial q_2} \delta q_2 + \frac{\partial z}{\partial q_3} \delta q_3
\]

where we require

\[
\frac{\partial x}{\partial t} \delta t = \frac{\partial y}{\partial t} \delta t = \frac{\partial z}{\partial t} \delta t = 0
\]

in order that we may ignore the forces of constraint. (We are making use of the concept of instantaneous, constrained virtual displacements.)

Using these definitions, the left side of D'Alembert's equation, in its formal representation, becomes

\[
\delta W = m \dot{x} \delta x + m \dot{y} \delta y + m \dot{z} \delta z
\]

\[
= m \dot{x} \left( \frac{\partial x}{\partial q_1} \delta q_1 + \frac{\partial x}{\partial q_2} \delta q_2 + \frac{\partial x}{\partial q_3} \delta q_3 \right) + m \dot{y} \left( \frac{\partial y}{\partial q_1} \delta q_1 + \frac{\partial y}{\partial q_2} \delta q_2 + \frac{\partial y}{\partial q_3} \delta q_3 \right) + m \dot{z} \left( \frac{\partial z}{\partial q_1} \delta q_1 + \frac{\partial z}{\partial q_2} \delta q_2 + \frac{\partial z}{\partial q_3} \delta q_3 \right)
\]

which is related to the time derivative of the kinetic energy. The right side of D'Alembert's equation becomes

\[
\delta W = F_x \delta x + F_y \delta y + F_z \delta z
\]

\[
= F_x \left( \frac{\partial x}{\partial q_1} \delta q_1 + \frac{\partial x}{\partial q_2} \delta q_2 + \frac{\partial x}{\partial q_3} \delta q_3 \right) + F_y \left( \frac{\partial y}{\partial q_1} \delta q_1 + \frac{\partial y}{\partial q_2} \delta q_2 + \frac{\partial y}{\partial q_3} \delta q_3 \right) + F_z \left( \frac{\partial z}{\partial q_1} \delta q_1 + \frac{\partial z}{\partial q_2} \delta q_2 + \frac{\partial z}{\partial q_3} \delta q_3 \right)
\]

This last equation is the defining equation for the generalized force \( F_{q_i} \) which is associated with the \( i^{th} \) generalized coordinate. Since the generalized coordinates are independent, we can write an expression like

\[
\delta W_{q_i} = m \left( \frac{\dot{x}}{\partial q_i} + \frac{\dot{y}}{\partial q_i} + \frac{\dot{z}}{\partial q_i} \right) \delta q_i = \left( F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i} \right) \delta q_i
\]

for each of the independent generalized coordinates.
In the next section, we will show that 

$$m \left( \dot{x} \frac{\partial x}{\partial q_i} + y \frac{\partial y}{\partial q_i} + z \frac{\partial z}{\partial q_i} \right) = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q_i}} \right) - \frac{\partial T}{\partial q_i}$$

(1.17)

where $T$ is the kinetic energy expressed in generalized coordinates, so that our virtual work equation becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q_i}} \right) - \frac{\partial T}{\partial q_i} = \left( F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i} \right) \delta q_i$$

(1.18)

where there are as many equations as there are degrees of freedom. Thus, for each degree of freedom, we have the equation

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q_i}} \right) - \frac{\partial T}{\partial q_i} = F_{q_i} \delta q_i$$

which is known as Lagrange's equation of motion.

**Derivation of the Left Side of D'Alembert's Equation**

D'Alembert's equation for virtual displacements can be written as a sum of terms like

$$\delta W_{q_i} = m \left( \dot{x} \frac{\partial x}{\partial q_i} + y \frac{\partial y}{\partial q_i} + z \frac{\partial z}{\partial q_i} \right) \delta q_i = \left( F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i} \right) \delta q_i$$

(1.19)

which is an expression for the virtual work associated with the $i^{th}$ component. As already demonstrated, the right hand side of this equation is typically written as $F_{q_i} \delta q_i$ and is called the generalized virtual work, with $F_{q_i}$ being the generalized force associated with the $q_i^{th}$ generalized coordinate. Likewise, the left hand side can be re-written in terms of the kinetic energy, as we will now demonstrate.

To get the left hand side in the right form, we will first examine the term

$$\dot{x} \frac{\partial x}{\partial q_i}$$

which can be written as

$$\dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \dot{x} \frac{\partial x}{\partial q_i} \right) - \dot{x} \frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right)$$

(1.20)

as is evident from the chain rule. Next, we look at the transformation equation

$$x = x(q_1, q_2, q_3, t)$$

(1.21)

and see that a time derivative of $x$ can be expressed as

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1} \dot{q_1} + \frac{\partial x}{\partial q_2} \dot{q_2} + \frac{\partial x}{\partial q_3} \dot{q_3} + \frac{\partial x}{\partial t}$$

(1.22)
Taking the partial derivative of $\dot{x}$ with respect to $\dot{q}_i$ in this last equation gives

$$\frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial x}{\partial q_i}$$

(1.23)

since we require all other variables to be fixed when we take the partial derivative. This equation shows that one can simply “add a dot to top and bottom” and maintain the equality when taking partial derivatives. This should be valid for any function of $q_1, q_2, q_3,$ and $t$. Finally, we want to show that

$$\frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{dx}{dt} \right)$$

(1.24)

To accomplish this, we again look at the equation for $dx/dt$ derived above. Now $x$ is just some function of the generalized coordinates, $x = x(q_1, q_2, q_3, t)$ and, in general, so is $\partial x/\partial q_i$. So, since

$$\frac{dx}{dt} = \frac{\partial x}{\partial q_1} \dot{q}_1 + \frac{\partial x}{\partial q_2} \dot{q}_2 + \frac{\partial x}{\partial q_3} \dot{q}_3 + \frac{\partial x}{\partial t}$$

by analogy we can write (substituting $\partial x/\partial q_i$ for $x$)

$$\frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right) = \frac{\partial}{\partial q_1} \left( \frac{\partial x}{\partial q_i} \right) \dot{q}_1 + \frac{\partial}{\partial q_2} \left( \frac{\partial x}{\partial q_i} \right) \dot{q}_2 + \frac{\partial}{\partial q_3} \left( \frac{\partial x}{\partial q_i} \right) \dot{q}_3 + \frac{\partial}{\partial t} \left( \frac{\partial x}{\partial q_i} \right)$$

(1.25)

Now, since the order of differentiation in a second partial is immaterial, this can be written

$$\frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left[ \left( \frac{\partial x}{\partial q_1} \right) \dot{q}_1 + \left( \frac{\partial x}{\partial q_2} \right) \dot{q}_2 + \left( \frac{\partial x}{\partial q_3} \right) \dot{q}_3 + \left( \frac{\partial x}{\partial t} \right) \right]$$

or

$$\frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{dx}{dt} \right)$$

(1.26)

which is what we wanted to show. This simply states that a total time derivative and a partial derivative with respect to a generalized coordinate can be interchanged, and since $x$ is just some function of $q_1, q_2, q_3$, and $t$, this last relationship should be valid for any function of the generalized coordinates and time.

Now we are ready to use all this to write our equation in standard form. The term $\dot{x} \frac{\partial x}{\partial q_i}$ can be written as

$$\dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \dot{x} \frac{\partial x}{\partial q_i} \right) - \dot{x} \frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right)$$
but since
\[
\frac{\partial \dot{x}}{\partial q_i} = \frac{\partial x}{\partial q_i}
\]
This can be written as
\[
\dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \dot{x} \frac{\partial x}{\partial q_i} \right) - \dot{x} \frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right)
\]
And, since
\[
\frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{dx}{dt} \right)
\]
this finally becomes
\[
\dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \dot{x} \frac{\partial x}{\partial q_i} \right) - \dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \dot{x} \frac{\partial x}{\partial q_i} \right) - \frac{d}{dt} \left( \frac{\partial x}{\partial q_i} \right)
\]
Terms like \( f \frac{d f}{dq} \) can be easily recognized as being related to the derivative of the square of \( f \), so we have
\[
\frac{\partial \dot{x}^2}{\partial q_i} = 2\dot{x} \frac{\partial \dot{x}}{\partial q_i} \quad \text{and} \quad \frac{\partial \dot{x}^2}{\partial q_i} = 2\dot{x} \frac{\partial \dot{x}}{\partial q_i}
\]
from which we obtain
\[
\dot{x} \frac{\partial x}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial \dot{x}}{\partial q_i} \right) - \frac{1}{2} \frac{\partial \dot{x}^2}{\partial q_i} = \left[ \frac{d}{dt} \left( \frac{\partial}{\partial q_i} \right) - \frac{\partial}{\partial q_i} \right] \frac{\dot{x}^2}{2}
\]
Thus,
\[
m \left( \dot{x} \frac{\partial x}{\partial q_i} + \frac{\partial \dot{y}}{\partial q_i} + \frac{\partial \dot{z}}{\partial q_i} \right) = \left\{ \left[ \frac{d}{dt} \left( \frac{\partial}{\partial q_i} \right) - \frac{\partial}{\partial q_i} \right] 1/2 m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right\}
\]
where \( T \) is the kinetic energy of the particle. Notice that the kinetic energy in the equation above is expressed in terms of the rectangular coordinates \( x, y, \) and \( z \). However, since the kinetic energy is a scalar quantity, it will have the same value expressed in any coordinate system. Thus, we must find the correct expression for the kinetic energy of the system and write this in terms of the generalized coordinates (and their derivatives) so we can apply this last equation.

**Applying Lagrange's Equations to a Simple System**

Once we have an appropriate expression for the kinetic energy \( T \) and for the generalized forces \( F_{q_i} \), we can, in principle, solve Lagrange's equation
for each of the generalized coordinates to obtain the equations of motion for each of the particles that make up the system. To illustrate how this is carried out in practice, we will again consider our simple example of an object of mass \( m \) moving on the inside of a spherical bowl (or, equivalently, a mass \( m \) hanging from an inextensible string of length \( R_o \) and allowed to move in two dimensions - the so-called spherical pendulum).

**Determination of the Kinetic Energy in Generalized Coordinates** Clearly, our first task is to write out an expression for the kinetic energy \( T \) in terms of the generalized coordinates and their derivatives.

**From the Transformation Equations** This can be accomplish in a very straightforward manner by making use of the transformation equations

\[
x = R_o \sin \theta \cos \phi \\
y = R_o \sin \theta \sin \phi \\
z = R_o |1 - \cos \theta|
\]

where we have already made use of the constraint equation

\[ R = R_o \]

The time derivatives of these equations give

\[
\dot{x} = R_o \cos \theta \dot{\theta} \cos \phi - R_o \sin \theta \sin \phi \dot{\phi} \\
\dot{y} = R_o \cos \theta \dot{\theta} \sin \phi + R_o \sin \theta \cos \phi \dot{\phi} \\
\dot{z} = + R_o \sin \theta \dot{\theta}
\]

Now, squaring these terms and adding we obtain, after some cancellation and recognizing that \( \sin^2 \theta + \cos^2 \theta = 1 \)

\[
T = \frac{1}{2} m \left( R_o^2 \dot{\theta}^2 + R_o^2 \sin^2 \theta \dot{\phi}^2 \right)
\]

**From the Vector Components** It is often simpler in practice to determine the kinetic energy term from the vector equation

\[
T = \frac{1}{2} m (\mathbf{\dot{V}} \cdot \mathbf{\ddot{V}})
\]

where the velocity vector is written in terms of the generalized coordinates. For example, the velocity vector in spherical coordinates can be written as

\[
\mathbf{\dot{V}} = \dot{R} \hat{r} + R \dot{\theta} \hat{\theta} + R \sin \theta \dot{\phi} \hat{\phi}
\]

where the first term is actually zero in our example because \( R \) is fixed. Taking the dot product, we obtain immediately the expression for the kinetic energy. In addition we can
determine the velocity from the equation
\[ |\vec{v}| = \frac{ds}{dt} \]
and in spherical coordinates
\[ ds^2 = (dR)^2 + (R d\theta)^2 + (R \sin \theta \, d\phi)^2 \]
from which we can immediately obtain the square of the velocity by simply replacing \( ds \) with \( \dot{s} \), \( dR \) with \( \dot{R} \), \( R \, d\theta \) with \( \dot{\theta} \), and \( R \sin \theta \, d\phi \) with \( \dot{\phi} \).

Once an expression for the kinetic energy in generalized coordinates is obtained, we next find the partial derivatives \( \partial T/\partial \dot{q}_i \) and \( \partial T/\partial \dot{q}_i \). Once this is done, we can take the total time derivative of the partial with respect to \( \dot{q}_i \), set it equal to the generalized force \( F_{\dot{q}_i} \), and solve Lagrange's equation for this particular generalized coordinate. Applying this to our example problem, we have
\[ T = \frac{1}{2} m \left( R_o^2 \dot{\theta}^2 + R_o^2 \sin^2 \theta \dot{\phi}^2 \right) \]
\[ \frac{\partial T}{\partial \dot{\theta}} = m R_o^2 \dot{\theta} \]
\[ \frac{\partial T}{\partial \dot{\phi}} = m R_o^2 \sin^2 \theta \dot{\phi} \]
Notice that the partial derivative of the kinetic energy with respect to the generalized coordinate \( \phi \) is zero. To understand the significance of this fact, we will look at the simple case of a single particle moving in one dimension (say along the \( x \)-axis). The kinetic energy for this case is simply
\[ T = \frac{1}{2} m \dot{x}^2 \]
The partial derivative with respect to \( \dot{x} \) is
\[ \frac{\partial T}{\partial \dot{x}} = m \dot{x} \]
which we immediately recognize as the momentum of the particle in the \( x \)-direction. Now, the fact that \( \partial T/\partial x = 0 \) means that the Lagrange equation reduces to
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = \ddot{x} = F_x \]
which is just Newton's second law in terms of the time derivative of the momentum.
Note: If the partial with respect to $x$ were not zero, then we would have
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = \dot{p}_x = F_x + \frac{\partial T}{\partial x}
\]
where we interpret the second term on the right as some type of pseudo-force. Thus, the fact that $\partial T/\partial q_i = 0$, implies that there are no pseudo-forces acting along this particular generalized coordinate.

An additional thing to point out here is the fact that the generalized force $F_x$ might also be zero, in which case we obtain
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = \dot{p}_x = 0 \Rightarrow p_x = \text{constant}
\]
which is a statement of the conservation of linear momentum. Thus, cases where the partial derivative of the kinetic energy with respect to a given generalized coordinate is zero often lead to conservation equations. We will have more to say about this later, when we discuss the form of Lagrange's equation for conservative systems.

**Determination of Generalized Forces**. Our next task is to determine the generalized forces. Again, we can take the straightforward approach of using that basic definition of the generalized force $F_{q_i}$ for a particular generalized coordinate $q_i$. The work done by the generalized force $F_{q_i}$ can be expressed as
\[
F_{q_i} \delta q_i = \left( F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i} \right) \delta q_i
\]
So our task is to evaluate the term in parentheses. For our example problem, the only forces acting on the mass $m$, other than constraint forces, is the force of gravity, given by
\[
\vec{F}_g = -mg \hat{k}
\]
and this force is only along the $z$-direction. This means that the generalized force equation simplifies to
\[
F_{q_i} = -mg \frac{\partial z}{\partial q_i}
\]
Since $z$ is a function only of $\theta$, this means that the generalized forces for our two independent coordinates is given by
\[
F_\theta = -mg R \sin \theta
\]
\[
F_\phi = 0
\]
Notice that this same result is obtained directly by simply realizing that the only work done by non-contact forces can be expressed as

\[ dW = -mg \, dz = -mg \left( \frac{\partial z}{\partial \theta} \, d\theta + \frac{\partial z}{\partial \phi} \, d\phi \right) = -mg \, R_o \, \sin \theta \, d\theta = F_\theta \, d\theta \]

which is essentially the process we followed in writing out the generalized forces.

**Solution to Lagrange's Equations** We are now in a position to solve Lagrange's equations for our example problem. The Lagrangian equation for each generalized coordinate \( (\theta, \phi) \) is

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = F_{q_i} \]

or

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = F_\theta \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \frac{\partial T}{\partial \phi} = F_\phi \]

where

\[ \frac{\partial T}{\partial \theta} = mR_o^2 \dot{\theta} \]
\[ \frac{\partial T}{\partial \phi} = mR_o^2 \sin^2 \theta \dot{\phi} \]
\[ \frac{\partial T}{\partial \theta} = mR_o^2 \sin \theta \cos \theta \dot{\phi}^2 \]
\[ \frac{\partial T}{\partial \phi} = 0 \]
\[ F_\theta = -mg \, R_o \, \sin \theta \]
\[ F_\phi = 0 \]

giving

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = F_\theta \Rightarrow \frac{d}{dt} \left( mR_o^2 \dot{\theta} \right) - mR_o^2 \sin \theta \cos \theta \dot{\phi}^2 = -mg \, R_o \, \sin \theta \]

and

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \frac{\partial T}{\partial \phi} = F_\phi \Rightarrow \frac{d}{dt} \left( mR_o^2 \sin^2 \theta \dot{\phi} \right) = 0 \]

This second equation is particularly important, because it indicates that the quantity \( mR_o^2 \sin^2 \theta \dot{\phi} \) is a constant. We pointed out earlier that the time derivative of \( \partial T/\partial \dot{x} \) was just the linear momentum \( p_i \). In this particular case, where both \( T \) and \( F_\phi \) are independent of the generalized coordinate \( \phi \), we see that the partial \( \partial T/\partial \dot{\phi} \) is a constant of the motion, and this turns out to be the angular momentum of the particle about the \( z \)-axis. So, defining

\[ L_z = I_{zz} \dot{\phi} = mR_o^2 \sin^2 \theta \dot{\phi} \]
the second Lagrangian equation can be written as
\[
\frac{d}{dt} L_z = 0 \Rightarrow L_z = mR_o^2 \sin^2 \theta \dot{\theta} = \text{constant}
\]

We can use this relationship to simplify the first Lagrangian equation (i.e., we can eliminate one of the variables in the first equation to obtain a differential equation in only one variable, \(\theta\)). So, starting with the first Lagrangian equation
\[
\frac{d}{dt} (mR_o^2 \theta) - mR_o^2 \sin \theta \cos \theta \dot{\phi}^2 = -mgR_o \sin \theta
\]
and substituting for \(\dot{\phi}\) using the conservation of the angular momentum \(L_z\), we have
\[
\frac{d}{dt} (mR_o^2 \theta) - mR_o^2 \sin \theta \cos \theta \left( \frac{L_z}{mR_o^2 \sin^2 \theta} \right)^2 = -mgR_o \sin \theta
\]
which further simplifies to give
\[
mR_o^2 \ddot{\theta} - \frac{L_z^2}{mR_o^2 \sin^3 \theta} \cos \theta = -mgR_o \sin \theta
\]
or, dividing by \(mR_o^2\) and putting all terms on one side of the equation, we obtain
\[
\ddot{\theta} + \left( \frac{g}{R_o} \right) \sin \theta - \frac{L_z^2}{m^2 R_o^2 \sin^3 \theta} \cos \theta = 0
\]
where \(L_z = mR_o^2 \sin^2 \theta \dot{\phi} = \text{constant}\)

Note: This last equation can be written in the form
\[
\ddot{\theta} + \left( \frac{g}{R_o} \right) \sin \theta - \frac{L_z^2}{2T_{zz}} \sin(2\theta) = 0
\]
where
\[
m(R_o \sin \theta)^2 = I_{zz}
\]
It is constructive at this point to consider some special cases. In the case where \(\dot{\phi} = 0\) (i.e., where \(\phi\) is a constant), the angular momentum is zero and the equation reduces to
\[
\ddot{\theta} + \left( \frac{g}{R_o} \right) \sin \theta = 0
\]
which is the equation for the simple pendulum, where motion is restricted to a plane. Another special case, is the one in which \(\theta\) is a constant, or \(\theta = \theta_o\) and \(\dot{\theta} = \dot{\theta} = 0\) giving
\[
\left( \frac{g}{R_o} \right) \sin \theta_o - \frac{L_z^2}{m^2 R_o^2 \sin^3 \theta_o} \cos \theta_o = 0
\]
and where
\[ \dot{L}_z = mR_o^2 \sin^2 \theta \dot{\phi} = \text{constant} \]
which gives (plugging the expression for \( L_z \) into the previous equation)
\[ \left( \frac{g}{R_o} \right) - \dot{\phi}^2 \cos \theta = 0 \]
or, solving for \( \dot{\phi} \), we obtain
\[ \dot{\phi}^2 = \frac{g}{R_o} \sec \theta = \text{constant} \]
This special situation is known as the conical pendulum.

To obtain a general, visual picture of the possible motions of this system, it is instructive to solve the differential equation for \( \theta(t) \) using MathCad and to plot the distance from the \( z \)-axis, \( \rho(t) \), and the angle \( \phi(t) \) on a polar plot and animate it. This means we will need to evaluate
\[ \rho(t) = R_o \sin \theta(t) \]
and
\[ \phi(t) = mR_o^2 \int \sin^2 \theta(t) \, dt = \int \frac{\sin^2 \theta(t)}{\theta} \, d\theta \]
and plot these in polar coordinates for different initial conditions. (This is done in the file "Spherical Pendulum.mcd" in the Advanced Mechanics Folder.)

**Lagrange's Equation for Conservative Forces**

In those cases where the force function can be written in terms of a potential energy function, we can further simplify the Lagrangian equations of motion. Beginning with Lagrange's equation for the \( q_i \text{th} \) generalized coordinate
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = F_{q_i} \]
If the generalized force \( F_{q_i} \) can be expressed as the derivative of some function \( U \), such that
\[ F_{q_i} = - \frac{\partial U}{\partial q_i} \]
we can write Lagrange's equation in the form
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial (T - U)}{\partial q_i} = 0 \]
Now, if the potential energy function is not a \textit{explicitly} a function of \( \dot{q}_i \), so that \( \partial U / \partial \dot{q}_i = 0 \), we can write

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0
\]

where \( L = T - U \) is the Lagrangian function.

For those cases where the generalized force is a combination of conservative and non-conservative forces, (i.e., where \( F_q = F_q' + f_q \), with \( F_q' \) being the conservative forces, and \( f_q \) the non-conservative forces), we can treat the conservative forces by introducing a potential energy function, and treat the non-conservative forces separately, giving the equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = f_q
\]

\textbf{Generalized Momenta and Ignorable Coordinates} For our example problem, the Lagrangian function would simply be

\[
L = T - U = \frac{1}{2} m \left( R^2 \dot{\theta}^2 + R^2 \sin^2 \phi \dot{\phi}^2 \right) - mg(R_o[1 - \cos \theta])
\]

from which we can obtain the equations of motion we have already derived. Notice, however, that since the Lagrangian is not a function of \( \phi \) explicitly, we can immediately obtain the conservation of angular momentum about the \( z \)-axis from

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{d}{dt} p_\phi = 0
\]

\[\Rightarrow p_\phi = mR_o^2 \sin^2 \theta \dot{\phi} = \text{constant}\]

Thus, whenever the forces considered are conservative, and the Lagrangian function \( (L = T - U) \) is not \textit{explicitly} a function of one of the generalized coordinates, say the \( q_j^{th} \), then the Lagrangian equation reduces to

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = 0
\]

which gives us a conservation of momentum relation

\[p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{constant}\]

We say that the generalized coordinate \( q_j \) is an \textbf{ignorable} or \textbf{cyclic} coordinate, and we define the quantity \( \partial L / \partial q_j \) as the \textbf{generalized momentum}, \( p_j \) which is \textbf{conjugate to} the generalized coordinate \( q_j \). It is, therefore, very important to notice those generalized coordinates which do \textit{not} appear explicitly in the Lagrangian function, for these coordinates are associated with a conserved momentum. Notice that the quantity that is conserved in our example problem is an angular momentum, since the generalized coordinate is an angle.
Lagrange's Equations for a System of Particles

The derivation of Lagrange's equations which we have carried out has been applied to the situation of a single particle under the influence of various forces. In the case where there are a number of particles connected by strings and constrained to move on various surfaces, we find that the equation for the kinetic energy is simply a combination of the kinetic energies for all of the individual particles (since kinetic energy is a scalar quantity). Likewise, the work done in moving a single particle some displacement $\delta q_i$ is also a scalar quantity. Thus, we will simply obtain the same mathematical expression for Lagrange's equation

$$\sum_{i=1}^{3N-n} \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} - F_{q_i} \right] \delta q_i = 0$$

where $N$ is the number of particles, and $n$ the number of constraint equations allowing us to reduce the number of coordinates to only those that are independent. However, $T$ is now the total kinetic energy of the entire system written in terms of all possible independent motions. Likewise, when determining the generalized force, $F_{q_i}$, one must consider all massed which move when the coordinate $q_i$ changes. To make this last point clear, consider the compound pulley system in the diagram below.

When $x_1$ changes, both masses $m_2$ and $m_3$ change their location, so the work done in changing $x_1$ must include the motion of both $m_1$ and $m_2$. (Remember that we assume all other independent coordinates are fixed during this virtual displacement, so that $s_2$ must remain fixed as you change $s_1$.)
Thus, Lagrange's equations are valid for a system of particles as well as a single particle. We must, however, exercise care when we express the kinetic energy and the generalized force.

**Forces of Constraint and Lagrange Multipliers**

Up until now, we have used the fact that the Lagrangian formalism allows us to ignore the forces of constraint to solve for the equations of motion. This is a tremendous simplification and allows us to solve some very difficult problems. However, an engineer must know these constraint forces in order to properly design the apparatus. We therefore need a technique which will allow us to determine the forces of constraint.

As a first step, let's examine the simple problem where a bead of mass \( m \) slides without friction on a wire which satisfies the equation constraint equation \( y = bx^2 \). We will assume that the mass moves in a constant gravitational field \( \vec{F}_g = mg \hat{y} \) and that an additional arbitrary applied force \( \vec{F}' \) also acts as shown on the diagram below.

![Diagram of bead sliding on wire with applied force](image)

If we approach this problem from the Newtonian perspective, we would write

\[
\begin{align*}
    m\ddot{x} &= F'_x + f_x \\
    m\ddot{y} &= F'_y + f_y - mg
\end{align*}
\]

where \( f_x \) and \( f_y \) are the forces of constraint due to the wire. Notice that in this formalism, we actually consider that the particle could move *arbitrarily* in the \( x \) and \( y \) direction *except for the effect of the constraint forces* \( f_x \) and \( f_y \). But if we *know* the equations of motion and the applied force \( \vec{F}' \), then we can solve these last two equations for the constraint forces.

Similarly, if we consider the possibility of the bead moving in *any arbitrary manner*, but include the forces of constraint *to make the bead move correctly*, we might
write the Lagrangian equation

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = F_{q_i} \]

where the generalized force \( F_{q_i} \) must include the forces of constraint. In the diagram above we might express the kinetic energy in terms of the generalized coordinates \( r \) and \( \theta \)

\[ T = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) \]

In order to determine the appropriate form of the generalized force, we must examine the virtual work done by all forces including the forces of constraint

\[ \delta W = F_r \, dr + F_\theta \, d\theta = \left[ F'_r + f_r - mg \sin \theta \right] \, dr + \left[ F'_\theta + f_\theta - mg \cos \theta \right] \, r \, d\theta \]

giving

\[ F_r = F'_r + f_r - mg \sin \theta \]
\[ F_\theta = r F'_\theta + r f_\theta - mgr \cos \theta \]

Lagrange's equations then become

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = F_r \]
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = F_\theta \]

or

\[ m \ddot{r} - mr \dot{\theta}^2 = F'_r + f_r - mg \sin \theta \]
\[ mr^2 \ddot{\theta} + 2m r \dot{r} \dot{\theta} = r F'_\theta + r f_\theta - mgr \cos \theta \]

Now, if we solve this problem using our earlier Lagrange techniques without constraint forces, we can actually solve for the equations of motion for \( r \) and \( \theta \). Knowing the equations of motion and the applied force \( \vec{F}' \), along with the force of gravity enables us to solve these last two equations for the constraint forces \( f_r \) and \( f_\theta \).

**Lagrange Multipliers and the Constraint Force** In actual practice, we will make use of the so-called Lagrange multipliers to determine the constraint forces. In this section we demonstrate how these two are related.

Recall that the equations of constraint are equations which relate the various coordinates and are often expressible in the form

\[ g(q_1, q_2, \ldots, q_n, t) = 0 \]

Constraint equations which can be expressed in terms of this last equation are called *holonomic* constraints. In many cases, we will have \( n \) coordinates and \( k \) equations of constraint which can be used to reduce the number of coordinates to \( n - k \) generalized,
independent coordinates. In some cases we cannot express the constraints as holonomic constraints, but, rather, must relate the various coordinates through differential changes, such as the constraint imposed when an object rolls without slipping

$$ds = r\, d\theta$$

We want to show, however, that constraint equations of the form

$$\sum_i A_i \dot{q}_i + B = 0$$

can be utilized just as holonomic constraint equations. (Note that this last equation is in the form of the constraint for rolling without slipping). If we associate $A_i$ with $\partial g/\partial q_i$ and $B$ with $\partial g/\partial t$, (where $g$ is just some arbitrary function of the $q_i$'s and time, this last equation becomes

$$\frac{dg}{dt} = \sum_i \frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial t} = 0$$

which is just the total time derivative of the function $g(q_1, q_2, \ldots, q_n, t)$, which implies that

$$g(q_1, q_2, \ldots, q_n, t) = C \Rightarrow g(q_1, q_2, \ldots, q_n, t) - C = 0$$

(where $C$ is just some constant) which is just the equation for holonomic constraints. Thus, constraints expressible in terms of the equation

$$\sum_i \frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial t} = 0$$

can be treated equivalently with holonomic constraints.

The change in $g$ can be expressed as

$$dg = \sum_i \frac{\partial g}{\partial q_i} \frac{dq_i}{dt} dt + \frac{\partial g}{\partial t} dt = 0$$

If we consider instantaneous, virtual displacements, this last equation would become

$$\delta g = \sum_i \frac{\partial g}{\partial q_i} \delta q_i + \frac{\partial g}{\partial t} \delta t = 0$$

where the time change is ignored to give

$$\delta g = \sum_i \frac{\partial g}{\partial q_i} \delta q_i = 0$$

Now, let's re-examine the example where the bead of mass $m$ slides on the wire $y = bx^2$. In this case, we assume that there are two coordinates that are independent, $q_1$ and $q_2$. To be valid, Lagrange's equations for these two coordinates should, therefore, contain the constraint forces. The general expression for D'Alembert's equation can be
written
\[
\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - F_{q_1} \right] \delta q_1 + \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} - F_{q_2} \right] \delta q_2 = 0
\]

But we know that the \( q_i \)'s are not really independent. There is an equation of constraint which relates the two. This equation can be expressed in the form
\[
g(q_1, q_2, t) = 0
\]
from which we can write
\[
dg = \frac{\partial g}{\partial q_1} \delta q_1 + \frac{\partial g}{\partial q_2} \delta q_2 = 0
\]
or
\[
\delta q_2 = - \frac{\partial g/\partial q_1}{\partial g/\partial q_2} \delta q_1
\]
This equation can be used to eliminate \( \delta q_2 \) from D'Alembert's equation to obtain
\[
\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - F_{q_1} \right] \delta q_1 - \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} - F_{q_2} \right] \frac{\partial g/\partial q_1}{\partial g/\partial q_2} \delta q_1 = 0
\]
so that for arbitrary changes in \( q_1 \) (not necessarily consistent with constraints) we have
\[
\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - F_{q_1} \right] \left( \frac{1}{\partial g/\partial q_1} \right) = \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} - F_{q_2} \right] \left( \frac{1}{\partial g/\partial q_2} \right)
\]
which is just the equation
\[
h_1(q_1, q_2, t) = h_2(q_1, q_2, t)
\]
which must be valid for all possible values of \( q_1, q_2, \) and time. This can, in general, only be true if each side is a constant, or perhaps some function of time, which we designate by \( \lambda(t) \), and which is known as the Lagrange undetermined multiplier. Thus, we have two equations
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - F_{q_1} = \lambda(t) \frac{\partial g}{\partial q_1}
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} - F_{q_2} = \lambda(t) \frac{\partial g}{\partial q_2}
\]
and the equation of constraint
\[ g(q_1, q_2, t) = 0 \]

Writing the two equations above in the form of D'Alembert's equation
\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial T}{\partial q_1} \right) - \frac{\partial T}{\partial q_1} &= F_{q_1} + f_{q_1} \\
\frac{d}{dt} \left( \frac{\partial T}{\partial q_2} \right) - \frac{\partial T}{\partial q_2} &= F_{q_2} + f_{q_2}
\end{align*}
\]
we immediately recognize that the generalized constraint force \( f_{q_1} \) must be given, in terms of the Lagrange multiplier, by the expression
\[ f_{q_1} = \lambda(t) \frac{\partial g}{\partial q_1} \]

When there are several equations of constraint, the generalized constraint force is given by
\[ f_{q_i} = \sum_j \lambda_j(t) \frac{\partial g_j}{\partial q_i} \]

For conservative systems, we can write equations in the form
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \lambda(t) \frac{\partial g}{\partial q_i}
\]

**An Example Problem** To illustrate these concepts, we will work the simple problem of a cylindrical disk rolling down an incline which makes an angle \( \alpha \) with the horizontal, as shown in the diagram below.

\[ \text{The two coordinates } y \text{ and } \theta \text{ are sufficient to describe the motion of the cylinder as it rolls down the plane. If the cylinder rolls without slipping, these two coordinates are} \]
related by the constraint equation

\[ dy = R \, d\theta \]

which we would normally use to eliminate one of the superfluous coordinates in favor of the other. However, we want to keep both as variables in this case so we can determine the constraint forces. The kinetic energy of the cylinder can be expressed as

\[ T = \frac{1}{2} m y'^2 + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} m y'^2 + \frac{1}{4} m R^2 \dot{\theta}^2 \]

To obtain the correct expression for the potential energy term you should examine the virtual work done on the cylinder as it moves. This is given by

\[ \delta W = mg \, \sin \alpha \, dy = F_y \, dy \]

which means that the potential energy function which is defined by the equation

\[ U = -mg \, \sin \alpha \]

is given by

\[ U = -mg \, y \sin \alpha + C_1 \]

where \( C_1 \) is a constant. Letting \( U = 0 \) at the point where \( y = 0 \), this constant must equal zero, so we have

\[ U = -mg \, y \sin \alpha \]

from which we obtain

\[ L = T - U = \frac{1}{2} m y'^2 + \frac{1}{4} m R^2 \dot{\theta}^2 + mg \, y \sin \alpha \]

To evaluate Lagrange's equations, we need the following

\[ \frac{\partial L}{\partial y'} = my \quad \frac{\partial L}{\partial y} = mg \sin \alpha \]

\[ \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m R^2 \dot{\theta} \quad \frac{\partial L}{\partial \theta} = 0 \]

In addition, we need the constraint equation

\[ dy = R \, d\theta \]

which we write in the form

\[ dy - R \, d\theta = 0 \]
This equation has the form
\[ \frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial \theta} \delta \theta = 0 \]

where we identify
\[ \frac{\partial g}{\partial y} = 1 \]
\[ \frac{\partial g}{\partial \theta} = -R \]

We are now in a position to write out Lagrange's equations with the undetermined multipliers
\[ \begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} &= \lambda(t) \frac{\partial g}{\partial y} \Rightarrow m\ddot{y} - mg \sin \alpha = \lambda(t) \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= \lambda(t) \frac{\partial g}{\partial \theta} \Rightarrow \frac{1}{2}mR^2\ddot{\theta} = -\lambda(t)R
\end{align*} \]

which we use in addition to the equation of constraint
\[ \dot{y} - R \dot{\theta} = 0 \]

to solve for the unknowns.

Using the constraint equation in the second of Lagrange's equations gives
\[ \frac{1}{2}mR^2 \left( \frac{1}{R} \ddot{y} \right) = -\lambda(t)R \Rightarrow \frac{1}{2}m\ddot{y} = -\lambda(t) \]

Solving for \( \ddot{y} \) and plugging into the first of Lagrange's equations gives
\[ -2\lambda(t) - mg \sin \alpha = \lambda(t) \Rightarrow \lambda(t) = -\frac{1}{3}mg \sin \alpha \]

which is just the generalized constraint force \( f_y \), and is the friction force acting on the cylinder to make it rotate. The generalized constraint force \( f_\theta \), given by
\[ f_\theta = -\lambda(t)R = \frac{1}{3}mgR \sin \alpha \]
is just the torque required to make the cylinder rotate about the center of mass. Using these generalized constraint forces, we can now solve for the equation of motion in the \( y \)-direction to obtain:
\[ m\ddot{y} = -\frac{1}{3}mg \sin \alpha + mgsin \alpha = \frac{2}{3}mg \sin \alpha \Rightarrow \ddot{y} = \frac{2}{3}mg \sin \alpha \]

Since the coordinate \( y \) and \( \theta \) are related by the constraint equation, the solution for the \( \theta \) motion is obvious.